

ON THE SPECTRUM OF C_0 -SEMIGROUPS

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ABSTRACT. In this paper we give characterizations of the spectrum of a C_0 -semigroup e^{At} in terms of certain solution properties of the differential equation $(*)$ $u' = Au + f$ and, in case X is a Hilbert space, also in terms of properties of $(\lambda - A)^{-1}$. We give several applications of these results including a study of the existence of dichotomic projections for $(*)$.

1. Introduction. Let X be a complex Banach space with norm $|\cdot|$ and A be a generator of a C_0 -semigroup e^{At} in X . One of the major open problems in the theory of such semigroups is the description of the spectrum $\sigma(e^{At})$ of e^{At} in terms of A . It is well known that $e^{\sigma(A)t} \subset \sigma(e^{At})$ holds for all generators A (cf. Hille and Phillips [12]), but also that the inclusion may be strict. In fact, Zabczyk [17] gave a nice example of a C_0 -group in a Hilbert space such that $|e^{At}| = e^{|t|}$ for all $t \in \mathbf{R}$ but $\sigma(A) \subset i\mathbf{R}$ is purely imaginary. Other examples may be found in Hille and Phillips [12] or Davies [6]. Of course, there are several large classes of generators A for which the spectrum of e^{At} can be expressed in terms of $\sigma(A)$. Some of these classes are:

- (i) A is bounded: $\sigma(e^{At}) = \exp(\sigma(A)t)$.
- (ii) X is a Hilbert space and A is normal: $\sigma(e^{At}) = \overline{\exp(\sigma(A)t)}$.
- (iii) e^{At} is continuous in $\mathbf{B}(X)$ for $t > t_0 \geq 0$: $\sigma(e^{At}) \setminus \{0\} = \exp(\sigma(A)t)$.

Here $\mathbf{B}(X)$ denotes the space of bounded linear operators in X normed in the usual way. In particular, (iii) includes C_0 -semigroups e^{At} such that e^{At} is compact for $t > t_0 \geq 0$ and C_0 -semigroups which are differentiable for $t > t_0 \geq 0$; see Davies [6].

Recently, Gearhart [8] obtained a characterization of $\sigma(e^{At})$ for semigroups of contractions in Hilbert spaces. His result is as follows:

$\mu \in \mathbf{C} \setminus \{0\}$ belongs to the resolvent set $\rho(e^{At}) \Leftrightarrow \{\lambda \in \mathbf{C}: e^{\lambda t} = \mu\} \subset \rho(A)$
and $(\lambda - A)^{-1}$ is uniformly bounded on this set.

However, his proof is quite lengthy and his arguments rely heavily on the harmonic analysis of contractions in Hilbert spaces developed by Foias and Sz. Nagy [7]. So the general case remained open, even in Hilbert spaces. Also, Gearhart's proof does not give much insight why the growth of $|(\lambda - A)^{-1}|$ on the solution set of $e^{\lambda t} = \mu$ determines whether $\mu \in \rho(e^{At})$ or not.

Received by the editors August 22, 1983 and, in revised form, December 5, 1983.

1980 *Mathematics Subject Classification.* 34G10, 47D05.

Key words and phrases. Semigroups, spectrum, periodic solutions, dichotomic projections.

¹Part of this research was done while the author was visiting the Department of Mathematics, Southern Illinois University at Carbondale.

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It is our first purpose in this paper to prove that Gearhart's result is valid for all C_0 -semigroups in Hilbert spaces. The key for our proof is the strong connection between the spectrum of e^{At} and periodic solutions of the inhomogeneous differential equation

$$(*) \quad u' = Au + f,$$

where f is a periodic forcing term. As a matter of fact, our proof is quite elementary and natural and gives a clear understanding of $\sigma(e^{At})$. As a byproduct we derive a complete characterization of the class of generators A such that for any given T -periodic f that equation has a unique T -periodic solution; cf. Theorem 1 below. This extends results obtained by Haraux [11].

A number of important consequences are then derived. We obtain a class of generators including (iii) mentioned above such that the relation $e^{\sigma(A)t} = \sigma(e^{At}) \setminus \{0\}$ holds in case X is a Hilbert space. We obtain a new simpler formula for the growth abscissa $\omega_0(A)$ of a C_0 -semigroup (see §3) and derive characterizations of the 'spectrum determined growth property' as well as of 'uniform asymptotic stability' in case X is a Hilbert space; see §3. The latter are very important for linear systems theory in infinite-dimensional spaces and have attracted many authors; see Curtain and Pritchard [4], Pritchard and Zabczyk [13], Slemrod [15] and the references given there.

In §4 we continue the analysis of the relationship between $\sigma(e^{At})$ and solution properties of the inhomogeneous differential equation (*). We show that there is dichotomic projection for e^{At} iff the unit sphere $\{|\mu| = 1\}$ is contained in the spectrum of e^A . Theorem 3 shows that this is equivalent to $i\mathbf{R} \subset \rho(A)$ and $\sup_{\mathbf{R}} |(i\tau - A)^{-1}| < \infty$ in case X is a Hilbert space. To our knowledge, such dichotomic projections have been obtained before only in case the essential spectrum of e^{At} is strictly contained in the unit circle, i.e. the unstable subspace is finite dimensional; for applications of this concept to functional differential equations with finite delays (e^{At} compact) see Hale [10], and for problems in age-dependent population dynamics (e^{At} noncompact) see Prüss [14] and Webb [16]. Once a dichotomic projection is known to exist, equation (*) behaves quite well; we show that precisely one bounded or bounded, uniformly continuous or almost periodic or convergent mild solution of (*) exists whenever f belongs to that class. Unfortunately, there is an example which shows that Theorem 2 is no longer valid in arbitrary Banach spaces; see Greiner, Voigt and Wolff [9, §4]. Hence boundedness of the resolvent of A is not sufficient to characterize the spectrum of e^{At} , in general, and therefore it would be interesting to know which additional properties of the resolvent of A are responsible for $\mu \in \mathbf{C}$ to belong to $\sigma(e^{At})$.

2. Periodic solutions. Let us fix some notation. If J is any real interval and $p \in [1, \infty)$, we let $L^p(J, X)$ denote the space of all strongly measurable functions $f: J \rightarrow X$ such that $\|f\|_p = (\int_J \|f(t)\|^p dt)^{1/p}$ is finite; $L^p(J, X)$ is a Banach space w.r.t. the norm $\|\cdot\|_p$. The space of bounded continuous functions $f: J \rightarrow X$ is denoted by $C(J, X)$ and its norm is $\|f\|_0 = \sup\{\|f(t)\|: t \in J\}$.

Recall that for $f \in L^1([0, T], X)$ given, a function $u \in C([0, T], X)$ is called a *mild solution* of

$$(1) \quad u'(t) = Au(t) + f(t), \quad t \in [0, T],$$

on $[0, T]$ with initial value $u_0 \in X$ if

$$(2) \quad u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)}f(s)ds$$

holds on $[0, T]$. Then $u(t)$ satisfies (1) for each $t \in [0, T]$ (u is a *strict solution*) iff $u \in C^1([0, T], X)$, i.e. u is continuously differentiable. If $u(t)$ is a mild solution on $[0, T]$ such that $u(0) = u(T)$, then it is clear that $u(t)$ can be continuously extended by periodicity to all of $\mathbf{R}$, hence is a T -periodic mild solution of (1) on $\mathbf{R}$, provided $f(t)$ has been extended T -periodically, too. Therefore, we call a mild solution of (1) on $[0, T]$ *T-periodic* if $u(0) = u(T)$.

Concerning the spectrum $\sigma(e^{At})$ of e^{At} , notice first that $\mu = e^{\lambda T} \in \mathbf{C}$ belongs to the resolvent set $\rho(e^{AT})$ iff $e^{\lambda T} - e^{AT} = e^{\lambda T}(I - e^{(A-\lambda)T})$ is invertible, i.e. iff $1 \in \rho(e^B)$ where $B = (A-\lambda)T$ is the generator of the C_0 -semigroup $e^{Bt} = e^{(A-\lambda)Tt}$. Therefore it suffices to consider ' $1 \in \rho(e^A)$ '. It is easy to see that there is a strong connection between the property ' $1 \in \rho(e^A)$ ' and 1-periodic solutions of (1). In fact, if $u(t)$ is a 1-periodic mild solution of (1) then (2) implies

$$(3) \quad (I - e^A)u(0) = \int_0^1 e^{A(1-s)}f(s)ds.$$

In particular, the 1-periodic solution of the homogeneous version of (1), i.e. $f(t) \equiv 0$, are precisely the functions $u(t) = e^{At}x$ with $x \in N(I - e^A)$, the kernel of $I - e^A$. These observations lead to

THEOREM 1. *Let X be a Banach space and e^{At} the C_0 -semigroup in X , generated by A . Then $1 \in \rho(e^A)$ iff for any $f \in C(J, X)$, $J = [0, 1]$, (1) admits precisely one 1-periodic mild solution.*

PROOF. ($\Rightarrow$) If $1 \in \rho(e^A)$ then $I - e^A$ is invertible, hence (3) determines the unique initial value of the 1-periodic mild solution $u(t)$ of (1) according to

$$(4) \quad u(0) = (I - e^A)^{-1} \int_0^1 e^{A(1-s)}f(s)ds.$$

($\Leftarrow$) Define $K: C(J, X) \rightarrow C(J, X)$ by means of $(Kf)(t) = u(t)$, where $u(t)$ denotes the unique 1-periodic mild solution of (1). Obviously, K is linear and everywhere defined; if $f_n \rightarrow f$ and $Kf_n \rightarrow u$ in $C(J, X)$ then it is clear from (2) that $u(t)$ is 1-periodic and a mild solution of (1), hence $u = Kf$ by uniqueness. This shows that the graph of K is closed, and so K is bounded, thanks to the Closed Graph Theorem. Now, consider $f(t) = e^{At}x$ for $x \in X$ and define $Sx = (Kf)(0)$. Of course, $S: X \rightarrow X$ is linear and bounded, and (3) yields

$$(I - e^A)(Sx + x) = \int_0^1 e^{A(1-s)}f(s)ds + x - e^Ax = x.$$

This shows that $I - e^A$ is surjective, and since $I - e^A$ is injective, too, we see that $I - e^A$ is invertible, i.e. $1 \in \rho(e^A)$. Q.E.D.

COROLLARY 1. *Let $1 \in \rho(e^A)$. Then we have:*

(i) *Equation (1) has a unique mild 1-periodic solution for any $f \in L^1(J, X)$, $J = [0, 1]$.*

(ii) Let $f \in W^{1,1}(J, X)$ or $f \in C(J, X) \cap L^1(J, Y)$, where $Y = D(A)$, normed by the graph norm of A . Then the 1-periodic mild solution is a strict solution.

The first part of Corollary 1 follows from the proof of Theorem 1 and the second part is a standard matter in semigroup theory since $(I - e^A)^{-1}$ commutes with A ; see [4]. If e^{At} is continuous in $\mathbf{B}(X)$ for $t > t_0 \geq 0$, then by (iii) of §1, the assumption ' $1 \in \rho(e^A)$ ' of Corollary 1 can be replaced by ' $\{2\pi in\}_{n \in \mathbf{Z}} \subset \rho(A)$ '.

Theorem 1 is the main tool for the proof of our central result on the spectrum of e^{At} in Hilbert space.

THEOREM 2. Let X be a Hilbert space and e^{At} the C_0 -semigroup in X generated by A . Then $1 \in \rho(e^A)$ iff

$$(5) \quad \{2\pi in\}_{n \in \mathbf{Z}} \subset \rho(A) \quad \text{and} \quad \sup_{n \in \mathbf{Z}} |(2\pi in - A)^{-1}| = M < \infty.$$

PROOF. ($\Rightarrow$) Let $1 \in \rho(e^A)$ and consider the operators T_n defined by

$$T_n x = (I - e^A)^{-1} \int_0^1 e^{As} e^{-2\pi in s} x \, ds, \quad x \in X, \quad n \in \mathbf{Z}.$$

Since A commutes with e^{As} as well as with $(I - e^A)^{-1}$, we have for $x \in D(A)$

$$\begin{aligned} (2\pi in - A)T_n x &= T_n(2\pi in - A)x = (I - e^A)^{-1} \int_0^1 (2\pi in - A)e^{(A-2\pi in)s} x \, ds \\ &= -(I - e^A)^{-1} \int_0^1 \frac{d}{ds} e^{(A-2\pi in)s} x \, ds = (I - e^A)^{-1}(x - e^A x) = x. \end{aligned}$$

Hence $2\pi in - A$ is invertible for each n and $(2\pi in - A)^{-1} = T_n$; we also obtain

$$|(2\pi in - A)^{-1}| = |T_n| \leq |(I - e^A)^{-1}| \sup\{|e^{As}|: 0 \leq s \leq 1\} = M < \infty,$$

i.e. (5) holds.

($\Leftarrow$) Suppose (5) holds and let $f \in C(J, X)$ be given, where $J = [0, 1]$. Then the Fourier-coefficients of f ,

$$f_n = \int_0^1 f(s) e^{-2\pi in s} \, ds, \quad n \in \mathbf{Z},$$

are well defined and satisfy Parseval's equality

$$\|f\|_2^2 = \int_0^1 |f(s)|^2 \, ds = \sum_{n \in \mathbf{Z}} |f_n|^2,$$

and

$$f(t) = \lim_{N \rightarrow \infty} \sum_{-N}^N f_n e^{2\pi in t} = \lim_{N \rightarrow \infty} f_N(t) \quad \text{in } L^2(J, X)$$

holds. If $u(t)$ is a mild 1-periodic solution of (1), then (2) implies that the Fourier-coefficients are

$$(6) \quad u_n = (2\pi in - A)^{-1} f_n, \quad n \in \mathbf{Z};$$

in particular, there is at most one 1-periodic mild solution. Conversely, let $u_N(t) = \sum_{-N}^N u_n e^{2\pi in t}$; then $u_N \in C^1(J, X)$ is 1-periodic and satisfies

$$(7) \quad u_N(t) = e^{At} u_N(0) + \int_0^t e^{A(t-s)} f_N(s) \, ds,$$

and u_N is even a strict solution. Now, (5) implies

$$\sum_{-\infty}^{\infty} |u_n|^2 \leq \sum_{-\infty}^{\infty} |(2\pi in - A)^{-1}|^2 |f_n|^2 \leq M^2 \sum_{-\infty}^{\infty} |f_n|^2 = M^2 \|f\|_2^2,$$

hence $u_N \rightarrow u$ in $L^2(J, X)$ as $N \rightarrow \infty$. Obviously,

$$\int_0^t e^{A(t-s)} f_N(s) ds \rightarrow \int_0^t e^{A(t-s)} f(s) ds \quad \text{as } N \rightarrow \infty,$$

uniformly for $t \in J$, and so $t = 1$ in (7) yields

$$(I - e^A)u_N(0) = \int_0^1 e^{A(1-s)} f_N(s) ds \rightarrow \int_0^1 e^{A(1-s)} f(s) ds.$$

On the other hand, multiplication of (7) by $e^{A(1-t)}$ and integration over J yields

$$e^A u_N(0) = \int_0^1 e^{A(1-t)} u_N(t) dt - \int_0^1 e^{A(1-t)} \int_0^t e^{A(t-s)} f_N(s) ds dt,$$

and the right-hand side of this equality converges as $N \rightarrow \infty$. This shows that $u_N(0) = (I - e^A)u_N(0) + e^A u_N(0)$ tends to some $u_0 \in X$. Finally, (7) implies that $u_N \rightarrow u$ in $C(J, X)$, and that $u(t)$ is a mild 1-periodic solution of (1). The proof is complete. Q.E.D.

Theorems 1 and 2 yield the following result on T -periodic solutions of (1) in Hilbert spaces, generalizing results in Haraux [11] who additionally had to assume

$$|(2\pi in - A)^{-1}| \leq C/|n| \quad \text{for all } n.$$

COROLLARY 2. *Let X be a Hilbert space and e^{At} a C_0 -semigroup in X . Then, for every $f \in L^1(J, X)$, (1) has a unique mild 1-periodic solution iff $\{(2\pi in)\}_{n \in \mathbf{Z}} \subset \rho(A)$ and $\sup_{n \in \mathbf{Z}} |(2\pi in - A)^{-1}| = M < \infty$.*

3. The spectrum of e^{At} . In view of the remark in front of Theorem 1, Theorem 2 immediately yields a characterization of $\rho(e^{At})$.

THEOREM 3. *Let X be a Hilbert space and e^{At} a C_0 -semigroup in X . Then $0 \neq \mu \in \rho(e^{At})$ iff $t^{-1} \operatorname{Log} \mu = \{\lambda \in \mathbf{C}: e^{\lambda t} = \mu\} \subset \rho(A)$ and $\sup\{ |(\lambda - A)^{-1}|: \lambda \in t^{-1} \operatorname{Log} \mu \} < \infty$.*

In this section we derive several consequences of Theorem 3 which illustrate its usefulness. Let us begin with a nice generalization of (iii) from §1.

PROPOSITION 1. *Let X be a Hilbert space and e^{At} be a C_0 -semigroup in X , and suppose A satisfies the following condition.*

(R) *There is $\psi: \mathbf{R}_+ \rightarrow \mathbf{R}_+$ increasing with $\psi(r) \rightarrow \infty$ as $r \rightarrow \infty$ such that $H_\psi = \{\lambda \in \mathbf{C}: \operatorname{Re} \lambda \geq -\psi(|\operatorname{Im} \lambda|)\} \subset \rho(A)$ and $\sup_{H_\psi} |(\lambda - A)^{-1}| = M < \infty$.*

Then $\sigma(e^{At}) \setminus \{0\} = e^{\sigma(A)t}$ holds for all $t \geq 0$.

PROOF. Let $\mu \in \mathbf{C} \setminus \{0\}$ be given and suppose that

$$t^{-1} \operatorname{Log} \mu = \{\lambda \in \mathbf{C}: e^{\lambda t} = \mu\} = \{t^{-1} \log \mu + t^{-1} 2\pi in: n \in \mathbf{Z}\} \subset \rho(A).$$

Condition (R) then implies that $(\lambda - A)^{-1}$ is uniformly bounded on $t^{-1} \operatorname{Log} \mu$, and so by Theorem 3 we obtain $\mu \in \rho(e^{At})$; this proves $\sigma(e^{At}) \setminus \{0\} \subset e^{\sigma(A)t}$. Since the converse inclusion always holds, we obtain the asserted equality. Q.E.D.

Note that C_0 -semigroups e^{At} which are continuous in $B(X)$ for $t > t_0 \geq 0$ satisfy

(R₁) *There is $\omega \in \mathbf{R}$ such that $\{\omega + i\tau\}_{\tau \in \mathbf{R}} \subset \rho(A)$ and $|(\omega + i\tau - A)^{-1}| \rightarrow 0$ as $|\tau| \rightarrow \infty$, and (R₁) in turn implies (R). In applications, Conditions (R) or (R₁) are easier to check than continuity of e^{At} in $\mathbf{B}(X)$ since, in general, information on the resolvent $(\lambda - A)^{-1}$ of A is better available. In order to see the limitations of (R), in case e^{At} is a C_0 -group and (R) holds then $\sigma(A)$ is bounded, hence A must be a bounded operator.*

For applications of semigroup theory to linear systems theory it is important to know whether e^{At} has the *spectrum determined growth property* (SDG), i.e. the *growth abscissa*

$$(8) \quad \omega_0(A) = \lim_{t \rightarrow \infty} t^{-1} \log |e^{At}| = \inf_{t > 0} t^{-1} \log |e^{At}|$$

of e^{At} and the *upper bound of the spectrum of A*

$$(9) \quad \omega_\sigma(A) = \sup\{\operatorname{Re} \lambda: \lambda \in \sigma(A)\}$$

coincide. It is well known that $e^{\omega_0(A)t}$ is the spectral radius of e^{At} and therefore the spectral containment $e^{\sigma(A)t} \subset \sigma(e^{At})$ implies that

$$(10) \quad \omega_\sigma(A) \leq \omega_0(A)$$

always holds. Theorem 3 yields a new formula for $\omega_0(A)$ which immediately leads to a characterization of those C_0 -semigroups in a Hilbert space which enjoy the SDG-property.

PROPOSITION 2. *Let X be a Hilbert space and e^{At} a C_0 -semigroup in X . Then*

$$(11) \quad \omega_0(A) = \inf\{\omega \geq \omega_0(A): |(\lambda - A)^{-1}| \leq M(\omega) \text{ for } \operatorname{Re} \lambda \geq \omega\}.$$

PROOF. Let ω_1 denote the right-hand side of (11); it is clear that $\omega_1 \leq \omega_0(A)$ holds, even in general Banach spaces. To prove the converse inequality, let $\mu \in \mathbf{C}$ be such that $|\mu| > e^{\omega_1}$. Then

$$\operatorname{Log} \mu = \{\log |\mu| + i \arg \mu + 2\pi i n: n \in \mathbf{Z}\} \subset \{\lambda \in \mathbf{C}: \operatorname{Re} \lambda \geq \operatorname{Log} |\mu| > \omega_1\} \subset \rho(A),$$

hence $(\lambda - A)^{-1}$ is uniformly bounded on $\operatorname{Log} \mu$. Theorem 3 implies $\mu \in \rho(e^A)$, and so $\sigma(e^A) \subset \{\mu \in \mathbf{C}: |\mu| \leq e^{\omega_1}\}$; this shows that $\omega_0(A) \leq \omega_1$, since $e^{\omega_0(A)}$ is the spectral radius of e^A . Q.E.D.

COROLLARY 3. *Let X be a Hilbert space. Then a C_0 -semigroup e^{At} in X has the SDG-property iff for each $\varepsilon > 0$ there is $M_\varepsilon \geq 1$ such that $|(\lambda - A)^{-1}| \leq M_\varepsilon$ for all $\operatorname{Re} \lambda \geq \omega_0(A) + \varepsilon$.*

Recall that the trivial solution $u(t) \equiv 0$ of

$$(12) \quad u'(t) = Au(t)$$

is said to be *uniformly asymptotically stable* if $e^{At}x \rightarrow 0$ as $t \rightarrow \infty$, uniformly w.r.t. $|x| \leq 1$. It is well known that this property is equivalent to exponential stability, i.e. $\omega_0(A) < 0$; see [4]. As another corollary to Proposition 2 we obtain

COROLLARY 4. *Let X be a Hilbert space and e^{At} be a C_0 -semigroup in X . Then $u(t) \equiv 0$ is uniformly asymptotically stable w.r.t. (12) iff $\{\lambda \in \mathbf{C}: \operatorname{Re} \lambda \geq 0\} \subset \rho(A)$ and there is $M \geq 1$ such that $|(\lambda - A)^{-1}| \leq M$ for all $\operatorname{Re} \lambda \geq 0$.*

4. Dichotomic projections. Recall the following definition of dichotomic projections for e^{At} .

DEFINITION. *A projection operator $P \in \mathbf{B}(X)$ is called a dichotomic projection for the C_0 -semigroup e^{At} in X if there are $M \geq 1$, $\delta > 0$ such that:*

- (P1) $Pe^{At} = e^{At}P$ for all $t \geq 0$;
- (P2) $|e^{At}Px| \leq Me^{-\delta t}|Px|$ for all $x \in X$, $t \geq 0$;
- (P3) $e^{At}(I - P)$ extends to a C_0 -group on $N(P)$;
- (P4) $|e^{At}(I - P)x| \leq Me^{\delta t}|(I - P)x|$ for all $x \in X$, $t \leq 0$.

In case A is a bounded operator it is known that e^{At} admits a dichotomic projection iff $i\mathbf{R} \subset \rho(A)$; see [5]. Many important properties of the inhomogeneous equation (1) can be derived if e^{At} is known to have a dichotomic projection, cf. [3, 5, 10] and the results below. It turns out that this concept is quite useful for unbounded operators, too, although its full strength will appear in connection with Theorems 2 and 3 in Hilbert spaces. The main result of this section is

THEOREM 4. *Let e^{At} be a C_0 -semigroup in the Banach space X . Then the following are equivalent:*

- (i) e^{At} admits a dichotomic projection.
- (ii) For each bounded $f \in C(\mathbf{R}, X)$ there is precisely one bounded mild solution $u \in C(\mathbf{R}, X)$ of (1).
- (iii) $S_1 = \{\mu \in \mathbf{C}: |\mu| = 1\} \subset \rho(e^A)$.

PROOF. (i) $\Rightarrow$ (ii). Let P be a dichotomic projection for e^{At} . Define Green's kernel $G_A(t)$ associated with (1) by means of

$$(13) \quad G_A(t) = \begin{cases} e^{At}P & \text{for } t > 0, \\ -e^{At}(I - P) & \text{for } t < 0; \end{cases}$$

recall that $e^{At}(I - P)$ has an extension to a C_0 -group on $N(P)$ which we denote by $e^{At}(I - P)$ again. In view of (P2) and (P4) we have

$$\int_{-\infty}^{\infty} |G_A(t)| dt \leq 2M/\delta < \infty$$

and therefore $K: BC(\mathbf{R}, X) \rightarrow BC(\mathbf{R}, X)$ given by

$$(14) \quad (Kf)(t) = (G_A * f)(t) = \int_{-\infty}^{\infty} G_A(t-s)f(s) ds$$

is well defined and continuous; here $BC(\mathbf{R}, X)$ denotes the space of bounded continuous functions $f: \mathbf{R} \rightarrow X$. We want to show that $u(t) = (Kf)(t)$ is the unique

bounded mild solution of (1). In fact, we have for $s < t$

$$\begin{aligned} u(t) - e^{A(t-s)}u(s) &= \int_{-\infty}^t e^{A(t-\tau)}Pf(\tau) d\tau - e^{A(t-s)} \int_{-\infty}^s e^{A(s-\tau)}Pf(\tau) d\tau \\ &\quad + e^{A(t-s)} \int_s^\infty e^{A(s-\tau)}Qf(\tau) d\tau - \int_t^\infty e^{A(t-\tau)}Qf(\tau) d\tau \\ &= \int_s^t e^{A(t-\tau)}Pf(\tau) d\tau + \int_s^t e^{A(t-\tau)}Qf(\tau) d\tau \\ &= \int_s^t e^{A(t-\tau)}f(\tau) d\tau, \end{aligned}$$

where $Q = I - P$, i.e. u is a mild solution of (1) on all of $\mathbf{R}$. To prove uniqueness suppose $u(t)$ is a bounded mild solution of the homogeneous equation (12). Then $u(t+s) = e^{At}u(s)$ for all $s \in \mathbf{R}$, $t \geq 0$; in particular, $u(s) = e^{At}u(s-t)$ for all $s \in \mathbf{R}$, $t \geq 0$. Hence

$$|Pu(s)| \leq |e^{At}P| |Pu(s-t)| \leq Me^{-\delta t} \cdot |P| \cdot |u|_0 \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

i.e. $Pu(s) \equiv 0$. Similarly, since $e^{At}Q$ is a C_0 -group on $N(P) = R(Q)$,

$$|Qu(s)| = |e^{-At}Qu(s+t)| \leq Me^{-\delta t}|Q||u|_0 \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

hence $Qu(s) \equiv 0$, i.e. $u(s) \equiv 0$.

(ii) $\Rightarrow$ (iii). Let $\mu = e^{i\alpha}$, $\alpha \in \mathbf{R}$, be given; we want to show that $e^{i\alpha} \in \rho(e^A)$, i.e. that $e^{i\alpha} - e^A = e^{i\alpha}(I - e^{A-i\alpha})$ is invertible. Since $v(t) = e^{-i\alpha t}u(t)$ is a mild solution of $v' = (A - i\alpha)v + f(t)e^{-i\alpha t}$ whenever $u(t)$ is a mild solution of (1) and conversely, it suffices to show that $1 \in \rho(e^A)$. In view of Theorem 1, it remains to prove that for a 1-periodic $f \in C(\mathbf{R}, X)$, the unique bounded solution $u(t)$ of (1) is 1-periodic. But this follows since $v(t) = u(t+1)$ is also a mild solution of (1), hence $u(t+1) \equiv u(t)$ by uniqueness.

(iii) $\Rightarrow$ (i). Let $S_1 \subset \rho(e^A)$; then

$$P = \frac{1}{2\pi i} \int_{|z|=1} (z - e^A)^{-1} dz$$

is well defined since $(z - e^A)^{-1}$ is holomorphic and bounded on an open neighbourhood of S_1 ; clearly P is linear and bounded, and the usual argument shows that $P^2 = P$, i.e. P is a projection. Since e^{At} commutes with $(z - e^A)^{-1}$ for all $t \geq 0$, $|z| = 1$, it is clear that $Pe^{At} = e^{At}P$, i.e. (P1) is satisfied; this also implies that e^{At} leaves $N(P)$ and $R(P)$ invariant. The C_0 -semigroup $U(t) = e^{At}|_{R(P)}$, the restriction of e^{At} to $R(P)$, then has spectrum $\sigma(U(1)) \subset \{\mu \in \mathbf{C}: |\mu| < 1\}$, hence the spectral radius of $U(1)$ is less than 1 and so there is $\delta > 0$ and $M \geq 1$ such that (P2) is fulfilled. Similarly, the C_0 -semigroup $V(t) = e^{At}|_{N(P)}$ has spectrum $\sigma(V(1)) \subset \{\mu \in \mathbf{C}: |\mu| > 1\}$, in particular $0 \in \rho(V(1))$. By Theorem 16.4.6 in [12], $V(t)$ extends to a C_0 -group in $N(P)$. Moreover, $\sigma(V(-1)) = \sigma(V(1)^{-1}) \subset \{\mu \in \mathbf{C}: |\mu| < 1\}$ and so there is $\delta > 0$ and $M \geq 1$ such that $|V(-t)| \leq Me^{-\delta t}$ for all $t \geq 0$, i.e. (P3) and (P4) follow. Q.E.D.

We want to stress that both spaces, $R(P)$ and $N(P)$ may be infinite dimensional. Note that in view of Theorem 3 we may add another equivalence in Theorem 4 in case X is a Hilbert space.

COROLLARY 5. Let X be a Hilbert space and e^{At} a C_0 -semigroup in X . Then (i)–(iii) of Theorem 4 are equivalent to

(iv) $i\mathbf{R} \subset \rho(A)$ and $\sup_{\mathbf{R}} |(i\tau - A)^{-1}| < \infty$.

In general, (iv) will be easier to check than (iii). Theorem 4 and Corollary 5 show that dichotomic projections for C_0 -semigroups e^{At} with unbounded A are almost as easy available as in the case of bounded A , at least in Hilbert spaces.

Before we prove some properties of the solution operator K of (1) defined by (14) we have to introduce some notation.

The translation group T_τ is defined by means of

$$(15) \quad (T_\tau f)(s) = f(\tau + s), \quad \tau, s \in \mathbf{R}.$$

T_τ is not of class C_0 in $BC(\mathbf{R}, X)$ but in each of the following subspaces of $BC(\mathbf{R}, X)$:

$$UBC = \{f \in BC(\mathbf{R}, X): f \text{ is uniformly continuous on } \mathbf{R}\},$$

$$C_1 = \left\{ f \in BC(\mathbf{R}, X): \lim_{t \rightarrow \pm\infty} f(t) = f(\pm\infty) \text{ exist} \right\},$$

$$C_0 = \{f \in C_1: f(\pm\infty) = 0\},$$

$$AP = \{f \in BC(\mathbf{R}, X): f \text{ is almost periodic (a.p.)}\};$$

recall that $f \in BC(\mathbf{R}, X)$ is called *almost periodic* if $(T_\tau f)_{\tau \in \mathbf{R}} \subset BC(\mathbf{R}, X)$ is relatively compact. If f is a.p. the *Bohr-transform*

$$(16) \quad a(\rho, f) = \lim_{N \rightarrow \infty} N^{-1} \int_0^N e^{-i\rho s} f(s) ds, \quad \rho \in \mathbf{R},$$

is well defined and the *exponent set* of f , $\exp(f) = \{\rho \in \mathbf{R}: a(\rho, f) \neq 0\}$, is at most countable; see [1].

PROPOSITION 3. Let e^{At} be a C_0 -semigroup in the Banach space X such that e^{At} admits a dichotomic projection P , and let K be defined by (13) and (14). Then K has the following properties.

(i) $KT_\tau = T_\tau K$ for all $\tau \in \mathbf{R}$;

(ii) K is leaving each of the spaces UBC , AP , C_1 , C_0 invariant;

(iii) if $f \in C_1$, then $\lim_{t \rightarrow \pm\infty} (Kf)(t) = -A^{-1}f(\pm\infty)$;

(iv) if $f \in AP$, then $\exp(Kf) = \exp(f)$ and

$$(17) \quad a(\rho, Kf) = (i\rho - A)^{-1}a(\rho, f) \quad \text{for all } \rho \in \mathbf{R}.$$

PROOF. Let $f \in BC(\mathbf{R}, X)$ and $\tau \in \mathbf{R}$; then $u(t) = (Kf)(t + \tau)$ is a mild solution of (1) with $f(t)$ replaced by $(T_\tau f)(t)$ and so (ii) of Theorem 4 implies $T_\tau Kf = u = KT_\tau f$ by uniqueness, i.e. (i) holds. This in turn yields

$$\|T_\tau Kf - Kf\|_0 = \|KT_\tau f - Kf\|_0 \leq \|K\| \cdot \|T_\tau f - f\|_0,$$

hence UBC is invariant w.r.t. K . Next, if $f \in AP$ then $\{T_\tau Kf\}_{\tau \in \mathbf{R}} = K\{T_\tau f\}_{\tau \in \mathbf{R}}$, hence $Kf \in AP$ since K is bounded. To prove assertion (iv), let $u(t) = (Kf)(t)$; then

$$e^{At}u(s) = u(t + s) - \int_s^{s+t} e^{A(s+t-\tau)} f(\tau) d\tau,$$

since u is a mild solution. Hence

$$\begin{aligned}
 e^{At}a(\rho, u) &= \lim_{N \rightarrow \infty} N^{-1} \int_0^N e^{At}u(s)e^{-i\rho s} ds \\
 &= e^{i\rho t} \lim_{N \rightarrow \infty} N^{-1} \int_0^N u(t+s)e^{-i\rho(t+s)} ds \\
 &\quad - \lim_{N \rightarrow \infty} N^{-1} \int_0^N e^{-i\rho s} \int_s^{s+t} e^{A(s+t-\tau)} f(\tau) d\tau ds \\
 &= e^{i\rho t}a(\rho, u) - \lim_{N \rightarrow \infty} N^{-1} \int_0^t e^{A(t-\tau)} e^{i\rho\tau} \int_0^N e^{-i\rho(s+\tau)} f(s+\tau) ds d\tau \\
 &= e^{i\rho t}a(\rho, u) - e^{i\rho t} \int_0^t e^{(A-i\rho)\tau} d\tau a(\rho, f),
 \end{aligned}$$

by Lebesgue's theorem, and therefore we obtain

$$t^{-1}(e^{At} - I)a(\rho, u) = t^{-1}(e^{i\rho t} - 1)a(\rho, u) - e^{i\rho t}t^{-1} \int_0^t e^{(A-i\rho)\tau} d\tau a(\rho, f).$$

Taking limits as $t \rightarrow 0^+$ we see that $a(\rho, u) \in D(A)$ for each $\rho \in \mathbf{R}$ and $(i\rho, A)a(\rho, u) = a(\rho, f)$, i.e. (iv) holds.

Finally, to show that the assertions for C_1 hold, notice first that K is continuous w.r.t. the topology of uniform convergence on compact subsets. Now let $f \in C_1$; then $T_\tau f \rightarrow f(\infty)$ as $\tau \rightarrow \infty$ uniformly on compact subsets, hence $T_\tau Kf = KT_\tau f \rightarrow Kf(\infty) = -A^{-1}f(\infty)$ uniformly on compact subsets, i.e. $(Kf)(t) \rightarrow -A^{-1}f(\infty)$ as $t \rightarrow \infty$. The same argument applies for $\tau \rightarrow -\infty$ and so $KC_1 \subset C_1$ and (iii) holds. Q.E.D.

ACKNOWLEDGEMENT. The author is indebted to R. Nagel for providing us with reference [9].

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